**oisson equation on Wasserstein space and

ffusion approximations for McKean-Vlaso

equation

Longjie Xie

Jiangsu Normal University

Joint work with Y. Li and F. Wu

the 17th Workshop on Markov Processes and Related Topic** Poisson equation on Wasserstein space and diffusion approximations for McKean-Vlasov equation

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Nov 27, 20 Longjie Xie (JSNU) Poisson equation and diffusion approximation Mov 27, 2022 2/19

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Consider the following two-time-scales stochastic system:

$$
dY_t^{\varepsilon} = F(X_{t/\varepsilon}, Y_t^{\varepsilon})dt + dW_t, \quad Y_0 = y \in \mathbb{R}^d,
$$
 (1.1)

OUTION:

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 $dY_t^\varepsilon = F(X_{t/\varepsilon}, Y_t^\varepsilon)dt + dW_t, \quad Y_0 = y \in \mathbb{R}^d,$
 $= (X_t)_{t \geq 0}$ is an ergodic Markov process possessing a unique

measure $\zeta(dx)$, and $0 < \varepsilon \ll 1$ represen where $X = (X_t)_{t \geq 0}$ is an ergodic Markov process possessing a unique invariant measure $\zeta(\mathrm{d}x)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

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- \triangleright Y_{t}^{ε} (slow variable): mathematical model for a phenomenon appearing at the natural time scale;
- $\triangleright X_{t/\varepsilon}$ (fast variable): fast random environment/effects at a faster time scale (with time order $1/\varepsilon$).

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Usually, system (1.1) is difficult to deal with due to the widely separated time scales. Thus a simplified equation which governs the evolution of the system over a long time scale (as $\varepsilon \to 0$) is highly desirable.

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Intuitively,

$$
X_{t/\varepsilon} \Rightarrow \zeta(\mathrm{d} x) \quad \text{as} \quad \varepsilon \to 0.
$$

9),
 $X_{t/\varepsilon} \Rightarrow \zeta(\mathrm{d}x)$ as $\varepsilon \to 0$.
 ι averaging the coefficient with respect to the fast variable,
 ιY_t^ε will converge to \bar{Y}_t , where
 $\mathrm{d}\, \bar{Y}_t = \bar{F}(\bar{Y}_t) \mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d$
 \bar Thus, by averaging the coefficient with respect to the fast variable, the slow part Y_{t}^{ε} will converge to \bar{Y}_{t} , where

$$
d\bar{Y}_t = \bar{F}(\bar{Y}_t)dt + dW_t, \quad Y_0 = y \in \mathbb{R}^d \tag{1.2}
$$

with

 $\bar{F}(y) := \int_{\mathbb{R}^d} F(x, y) \zeta(\textup{d} x).$

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 $\mathrm{d}\, \bar{Y}_t = \bar{F}(\bar{Y}_t) \mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d$
 $\bar{F}($ This theory, known as the averaging principle, was first developed by Bogolyubov (1937) for ODEs and extended to the SDEs by Khasminskii (1966).

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

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\nRefer the following fast-slow stochastic system in
$$
\mathbb{R}^{d_1+d_2}
$$
:

\n
$$
\begin{cases}\ndX_t^{\varepsilon} = \varepsilon^{-1}b(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \varepsilon^{-1/2}dW_t^1, & X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \\
dY_t^{\varepsilon} = F(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + dW_t^2, & Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}, \\
0 < \varepsilon \ll 1 \text{ is a small parameter.}\n\end{cases}
$$
\n(1.3)

\nLongije Xie (JSNU)

\nPoisson equation and diffusion approximation

\nNo: 27, 2022

where $0 < \varepsilon \ll 1$ is a small parameter.

 QQ

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

[D](#page-0-0)raft (dX ε ^t = ε −1 b(X ε t , Y ε t)dt + ε [−]1/2dW ¹ t , X ε ⁰ = x ∈ R d1 , dY ε ^t = F(X ε t , Y ε t)dt + dW ² t , Y ε ⁰ = y ∈ R d2 , (1.3)

where $0 < \varepsilon \ll 1$ is a small parameter.

The intuitive idea for deriving a simplified equation for (1.3) is based on the observation that:

- \circ during the fast transients, the slow variable remains "constant";
- \Diamond by the time its changes become noticeable, the fast variable has almost reached its quasi-steady state.

ound - Averaging principle

he natural time scaling $t \mapsto \varepsilon t$, the process $\tilde{X}_t^{\varepsilon} := X_{\varepsilon t}^{\varepsilon}$ satis
 $\mathrm{d}\tilde{X}_t^{\varepsilon} = b(\tilde{X}_t^{\varepsilon}, Y_{\varepsilon t}^{\varepsilon}) \mathrm{d}t + \mathrm{d}\tilde{W}_t^1,$
 $\mathcal{U}_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is \diamond With the natural time scaling $t\mapsto \varepsilon t$, the process $\tilde X^\varepsilon_t:=X^\varepsilon_{\varepsilon t}$ satisfies

$$
\mathrm{d}\tilde{X}_{t}^{\varepsilon}=b(\tilde{X}_{t}^{\varepsilon},Y_{\varepsilon t}^{\varepsilon})\mathrm{d}t+\mathrm{d}\tilde{W}_{t}^{1},
$$

where $\tilde{W}_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is a new BM.

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 $d\tilde{X}_t^{\varepsilon} = b(\tilde{X}_t^{\varepsilon}, Y_{\varepsilon t}^{\varepsilon}) dt + d\tilde{W}_t^1$,
 $\gamma_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is a new BM.

ne Thus we need to consider the auxiliary process $X_t^{\mathcal{Y}}$ which satisfies the frozen equation

$$
dX_t^y = b(X_t^y, y)dt + dW_t^1, \quad X_0^y = x \in \mathbb{R}^{d_1}.
$$

Under certain recurrence conditions, the process $X_t^{\mathcal{Y}}$ process a unique invariant measure $\zeta^{\mathbf{y}}(\mathrm{d} \mathbf{x})$.

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ound - Averaging principle

by averaging the coefficients with respect to parameter in fa

the slow part Y_t^{ε} will converge to \tilde{Y}_t , where
 $\mathrm{d}\, \bar{Y}_t = \bar{F}(\bar{Y}_t) \mathrm{d} t + \mathrm{d} W_t^2$
 $\bar{F}(y) := \int_{\mathbb{R}^{d_1}} F(x$ \circ Then by averaging the coefficients with respect to parameter in fast variable, the slow part Y_{t}^{ε} will converge to \bar{Y}_{t} , where

$$
d\bar{Y}_t = \bar{F}(\bar{Y}_t)dt + dW_t^2
$$
 (1.4)

with

$$
\bar{F}(y) := \int_{\mathbb{R}^{d_1}} F(x, y) \zeta^y(\mathrm{d} x).
$$

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

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\nler the following multiscale SDE in
$$
\mathbb{R}^{d_1+d_2}
$$
:

\n
$$
\begin{cases}\ndX_t^{\varepsilon} = \varepsilon^{-1} c(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-2} b(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-1} dW_t^1, \\
dY_t^{\varepsilon} = \varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + F(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + dW_t^2, \\
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}.\n\end{cases}
$$
\n(1.5)

\nngjie Xie (JSNU) Poisson equation and diffusion approximation

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Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

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\n
$$
dX_t^{\varepsilon} = \varepsilon^{-1} c(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-2} b(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-1} dW_t^1,
$$
\n
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dY_t^{\varepsilon} = \varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + F(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + dW_t^2,
$$
\n(1.5)\n
$$
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}.
$$
\netc. dX_t^{ε} is a fast, X_t^{ε} ;

\n dX_t^{ε} is a the flow process Y_t^{ε} has a fast varying component, which is now to be closely related to the homogenization in PDEs.\nNow to be closely related to the homogenization in PDEs.

\nFigure Xie (JSNU) Poisson equation and diffusion approximation. Now 27, 2022 8 / 2.2022 8 / 2.2022 8.

 \diamond there exist two time scales in the fast motion X^ε_t ,

 \diamond even the slow process Y^ε_t has a fast varying component, which is known to be closely related to the homogenization in PDEs.

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

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$$
dX_t^{\varepsilon} = \varepsilon^{-1} c(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-2} b(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-1} dW_t^1,
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dY_t^{\varepsilon} = \varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + F(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + dW_t^2,
$$
\n(1.5)\n
$$
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}.
$$
\n
$$
e^{-\varepsilon} = \varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-1} dW_t^2,
$$
\n
$$
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$$
\n
$$
e^{-\varepsilon^{-1} H(X_t^{\varepsilon})}
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\n
$$
e^{-\varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon})} = \varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \varepsilon^{-1} dW_t^1,
$$
\n
$$
e^{-\varepsilon^{-1} H(X_t^{\varepsilon})}
$$
\n
$$
e^{-\varepsilon^{-1} H(X_t^{\varepsilon}, Y_t^{\varepsilon})} dt + \varepsilon^{-2} h(X_t^{\varepsilon}, Y_t^{\varepsilon}) dt + \vare
$$

- \diamond there exist two time scales in the fast motion X^ε_t ,
- \diamond even the slow process Y^ε_t has a fast varying component, which is known to be closely related to the homogenization in PDEs.

History results:

- Papanicolaou, Stroock and Varadhan (1976);
- • Pardoux and Veretennikov (2001, 03, 05).

We need to consider the frozen equation

$$
dX_t^y = b(X_t^y, y)dt + dW_t^1, \quad X_0^y = x \in \mathbb{R}^{d_1}
$$

and the corresponding Poisson equation in \mathbb{R}^{d_1} :

$$
\mathscr{L}_0(x,y)\Phi(x,y)=-H(x,y),\quad x\in\mathbb{R}^{d_1},\qquad(1.6)
$$

where $y\in\mathbb{R}^{d_2}$ is a parameter, and $\mathscr{L}_0(x,y)$ is given by

1. Consider the frozen equation

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$$
dX_{t}^{y} = b(X_{t}^{y}, y)dt + dW_{t}^{1}, \quad X_{0}^{y} = x \in \mathbb{R}^{d_{1}}
$$
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$$
\n
$$
\in \mathbb{R}^{d_{2}} \text{ is a parameter, and } \mathscr{L}_{0}(x, y) \text{ is given by}
$$
\n
$$
\mathscr{L}_{0}(x, y) := \frac{1}{2}\Delta_{x} + b(x, y) \cdot \nabla_{x}.
$$
\nExercise (JSNU)

\nPoisson equation and diffusion approximation

\nNo 27, 20

As $\varepsilon \to 0$, the slow process Y_{t}^{ε} will converge to \bar{Y}_{t} with

$$
d\bar{Y}_t = \bar{F}(\bar{Y}_t)dt + \left(\overline{c \cdot \nabla_x \Phi}(\bar{Y}_t) + \overline{H \cdot \nabla_y \Phi}(\bar{Y}_t)\right)dt + \left(\mathbb{I} + \sqrt{\overline{H * \Phi}}(\bar{Y}_t)\right)dW_t^2,
$$

where the new drift and the diffusion coefficients are given by

0, the slow process
$$
Y_t^{\varepsilon}
$$
 will converge to \overline{Y}_t with

\n
$$
d\overline{Y}_t = \overline{F}(\overline{Y}_t)dt + \left(\overline{c \cdot \nabla_x \Phi}(\overline{Y}_t) + \overline{H \cdot \nabla_y \Phi}(\overline{Y}_t)\right)dt + \left(\mathbb{I} + \sqrt{\overline{H * \Phi}}(\overline{Y}_t)\right)dW_t^2,
$$
\ne new drift and the diffusion coefficients are given by

\n
$$
\overline{c \cdot \nabla_x \Phi}(y) := \int_{\mathbb{R}^{d_1}} c(x, y) \cdot \nabla_x \Phi(x, y) \zeta^y(dx),
$$
\n
$$
\overline{H \cdot \nabla_y \Phi}(y) := \int_{\mathbb{R}^{d_1}} H(x, y) \cdot \nabla_y \Phi(x, y) \zeta^y(dx),
$$
\n
$$
\overline{H * \Phi}(y) := \int_{\mathbb{R}^{d_1}} H(x, y) * \Phi(x, y) \zeta^y(dx).
$$
\ne Xie (JSNU)

\nPoisson equation and diffusion approximation

Bezemek and Spiliopoulos (2022) considered the multiscale distribution dependent SDE in $\mathbb{R}^{d_1+d_2}$:

\n
$$
\mathbf{c}
$$
 is the function of the function \mathbf{c} and \mathbf{c} is the function $\int dX_t^\varepsilon = \varepsilon^{-1} c(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) dt + \varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) dt + \varepsilon^{-1} dW_t^1,$ \n $\int dY_t^\varepsilon = \varepsilon^{-1} H(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) dt + F(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon}) dt + dW_t^2,$ \n $\int dY_t^\varepsilon$ \n

\n\n \mathbf{c} is the distribution of the slow process Y_t^ε .\n

\n\n \mathbf{c} is the function of \mathbf{c} and \mathbf{c} .\n

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where $\mathcal{L}_{Y_t^{\varepsilon}}$ is the distribution of the slow process $Y_t^{\varepsilon}.$

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\begin{aligned}\n &\text{cl}\ \mathsf{ex} &\text{cl}\ \mathsf{ex} &\text{cl}\ \mathsf{ex} &\text{cl}\ \mathsf{ex} \\
 &\text{cl}\ \mathsf{ex} &\text{cl}\ \mathsf
$$

where $\mathcal{L}_{Y_t^{\varepsilon}}$ is the distribution of the slow process $Y_t^{\varepsilon}.$

The frozen equation is still given by

$$
dX_t^y = b(X_t^y, y, \mu)dt + dW_t^1
$$

and the corresponding Poisson equation in \mathbb{R}^{d_1} is:

$$
\mathscr{L}_0(x, y, \mu)\Phi(x, y, \mu) = -H(x, y, \mu), \qquad (1.8)
$$

where $(y,\mu)\in\mathbb{R}^{d_2}\times\mathscr{P}(\mathbb{R}^{d_2})$ are parameters, and $\mathscr{L}_0(x,y,\mu)$ is given by

$$
\mathscr{L}_0(x,y,\mu) := \frac{1}{2}\Delta_x + b(x,y,\mu) \cdot \nabla_x.
$$

As $\varepsilon \to 0$, the slow process Y_{t}^{ε} will converge to \bar{Y}_{t} with

$$
d\bar{Y}_t = \bar{F}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})dt + \left(\overline{c \cdot \nabla_x \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}) + \overline{H \cdot \nabla_y \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\right)dt + \left(\mathbb{I} + \sqrt{\overline{H * \Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\right)dW_t^2,
$$

where the new drift and the diffusion coefficients are given by

0, the slow process
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 will converge to \overline{Y}_t with

\n
$$
= \overline{F}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t}) dt + \left(\overline{c} \cdot \nabla_x \Phi(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t}) + \overline{H} \cdot \nabla_y \Phi(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})\right) + \left(\mathbb{I} + \sqrt{\overline{H} \ast \Phi}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})\right) dW_t^2,
$$
\ne new drift and the diffusion coefficients are given by

\n
$$
\overline{c} \cdot \nabla_x \Phi(y, \mu) := \int_{\mathbb{R}^{d_1}} c(x, y, \mu) \cdot \nabla_x \Phi(x, y, \mu) \zeta^{y, \mu}(dx),
$$
\n
$$
\overline{H} \cdot \nabla_y \Phi(y, \mu) := \int_{\mathbb{R}^{d_1}} H(x, y, \mu) \cdot \nabla_y \Phi(x, y, \mu) \zeta^{y, \mu}(dx),
$$
\n
$$
\overline{H \ast \Phi}(y, \mu) := \int_{\mathbb{R}^{d_1}} H(x, y, \mu) \ast \Phi(x, y, \mu) \zeta^{y, \mu}(dx).
$$
\ne Xie (JSNU)

\nPoisson equation and diffusion approximation

Main result

Consider the multiscale distribution dependent SDE in $\mathbb{R}^{d_1+d_2}$:

Again result

\nConsider the multiscale distribution dependent SDE in
$$
\mathbb{R}^{d_1+d_2}
$$
:

\n
$$
\begin{cases}\ndX_t^{\varepsilon} = \varepsilon^{-1} c(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}}) dt + \varepsilon^{-2} b(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, \mathcal{L}_{Y_t^{\varepsilon}}) dt + \varepsilon^{-1} dW_t^1, \\
dY_t^{\varepsilon} = \varepsilon^{-1} H(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}}) dt + F(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}}) dt + dW_t^2, \\
\text{where } \mathcal{L}_{X_t^{\varepsilon}} \text{ and } \mathcal{L}_{Y_t^{\varepsilon}} \text{ are the distribution of the processes } X_t^{\varepsilon} \text{ and } Y_t^{\varepsilon}, \\
\text{respectively.}
$$
\nUsing the following equation, and differentiation approximation, we have

\n
$$
\text{Using } X_t^{\varepsilon} \text{ is the } \mathbb{R} \to \mathbb{R} \to
$$

where $\mathcal{L}_{X_t^{\varepsilon}}$ and $\mathcal{L}_{Y_t^{\varepsilon}}$ are the distribution of the processes X_t^{ε} and $Y_t^{\varepsilon},$ respectively.

Main result

Consider the multiscale distribution dependent SDE in $\mathbb{R}^{d_1+d_2}$:

Again result

\nConsider the multiscale distribution dependent SDE in
$$
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:

\n
$$
\begin{cases}\ndX_t^{\varepsilon} = \varepsilon^{-1}c(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}})dt + \varepsilon^{-2}b(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, \mathcal{L}_{Y_t^{\varepsilon}})dt + \varepsilon^{-1}dW_t^1, \\
dY_t^{\varepsilon} = \varepsilon^{-1}H(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}})dt + F(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}})dt + dW_t^2, \\
A_t^{\varepsilon} = \varepsilon^{-1}H(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}})dt + F(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon}, \mathcal{L}_{Y_t^{\varepsilon}})dt + dW_t^2, \\
A_t^{\varepsilon} = \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, Y_t^{\varepsilon})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E_t(X_t^{\varepsilon}, \mathcal{L}_{X_t^{\varepsilon}})dt + \varepsilon^{-1}E
$$

where $\mathcal{L}_{X_t^{\varepsilon}}$ and $\mathcal{L}_{Y_t^{\varepsilon}}$ are the distribution of the processes X_t^{ε} and $Y_t^{\varepsilon},$ respectively.

Now, we should consider the following frozen equation

$$
dX_t^{\mu} = b(X_t^{\mu}, \mathcal{L}_{X_t^{\mu}}, \mu)dt + dW_t^1,
$$

which is a McKean-Vlasov type equation, and the corresponding Poisson equation in $\mathbb{R}^{d_1}\times\mathscr{P}(\mathbb{R}^{d_1})$:

$$
\mathscr{L}_0(x,\nu,\mu)\Phi(x,\nu,y,\mu)=-H(x,\nu,y,\mu),\qquad(2.10)
$$

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where the $\mathscr{L}_0(x,\nu,\mu)$ is given by

result

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$$
\mathcal{L}_0(x, \nu, \mu)\varphi(x, \nu)
$$
 is given by

\n
$$
\mathcal{L}_0(x, \nu, \mu)\varphi(x, \nu)
$$

\n∴
$$
\frac{1}{2}\Delta_x\varphi(x, \nu) + b(x, \nu, \mu) \cdot \nabla_x\varphi(x, \nu)
$$

\n+
$$
\int_{\mathbb{R}^{d_2}} \left[b(\tilde{x}, \nu, \mu) \cdot \partial_\nu\varphi(x, \nu)(\tilde{x}) + \frac{1}{2}\partial_{\tilde{x}} \left[\partial_\nu\varphi(x, \nu)(\tilde{x}) \right] \right] \nu(\mathrm{d}\tilde{x})
$$
.

\nLet part of the operator involves the usual derivatives in the spa-
e (which is standard).

\ntegral part contains the Lions derivative of the test function w to the measure argument.

\nFigure Xie (JSNU)

\nPoisson equation and diffusion approximation

\nNo 27, 2022

The first part of the operator involves the usual derivatives in the space variable (which is standard).

The integral part contains the Lions derivative of the test function with respect to the measure argument.

Main result

As $\varepsilon \to 0$, the slow process Y_{t}^{ε} will converge to \bar{Y}_{t} with

$$
\begin{aligned}\n\textbf{n} \text{ result} \\
\Rightarrow \text{ 0, the slow process } Y_t^{\varepsilon} \text{ will converge to } \bar{Y}_t \text{ with} \\
\mathrm{d}\,\bar{Y}_t &= \bar{F}\big(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}\big)\mathrm{d}t + \Big(\overline{c \cdot \nabla_x \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}\big) + \overline{H \cdot \nabla_y \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}\big)\Big)\mathrm{d}t \\
&\quad + \sqrt{\mathbb{I} + \overline{H \ast \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})}\mathrm{d}W_t^2 \\
&\quad + \underline{\tilde{\mathbb{E}}}\bigg(\overline{\overline{c \cdot \partial_\nu \Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\big(\tilde{\bar{Y}}_t\big)\big)\mathrm{d}t, \\
\text{re } \tilde{Y}_t \text{ is a copy of } \bar{Y}_t \text{ defined on a copy } (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}) \text{ of the original} \\
\text{ability space, and } \tilde{\mathbb{E}} \text{ is the expectation taken with respect to } \tilde{\mathbb{P}}.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n\text{Longile Xie (JSNU)} \qquad \text{Poisson equation and diffusion approximation} \qquad \text{Nov 27, 2022}\n\end{aligned}
$$

where $\tilde{\bar{Y}}_t$ is a copy of \bar{Y}_t defined on a copy $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of the original probability space, and $\mathbb{\tilde{E}}$ is the expectation taken with respect to $\mathbb{\tilde{P}}$.

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Main result

As $\varepsilon \to 0$, the slow process Y_{t}^{ε} will converge to \bar{Y}_{t} with

In result
\n⇒ 0, the slow process
$$
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$$
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\n
$$
d\overline{Y}_t = \overline{F}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})dt + (\overline{c \cdot \nabla_x \Phi}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t}) + \overline{H \cdot \nabla_y \Phi}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t}))dt + \sqrt{\mathbb{I} + \overline{H * \Phi}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})}dW_t^2 + \tilde{\mathbb{E}}(\overline{\overline{c \cdot \partial_v \Phi}}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})(\overline{Y}_t))}dt,
$$
\nre $\overline{\tilde{Y}}_t$ is a copy of \overline{Y}_t defined on a copy $(\overline{\tilde{\Omega}}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of the original
\nability space, and $\tilde{\mathbb{E}}$ is the expectation taken with respect to $\tilde{\mathbb{P}}$.
\nblue part can be written as
\n
$$
\tilde{\mathbb{E}}(\overline{\overline{c \cdot \partial_v \Phi}}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})(\overline{\tilde{Y}}_t)) = \int_{\mathbb{R}^{d_2}} \overline{\overline{c \cdot \partial_v \Phi}}(\overline{Y}_t, \mathcal{L}_{\overline{Y}_t})(\tilde{Y})\mathcal{L}_{\overline{Y}_t}(d\tilde{Y}),
$$
\nthe depends on \overline{Y}_t and its distribution $\mathcal{L}_{\overline{Y}_t}$.
\nLongile Xie (JSNU) Poisson equation and diffusion approximation
\nNow 27, 2022

where $\tilde{\bar{Y}}_t$ is a copy of \bar{Y}_t defined on a copy $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of the original probability space, and $\mathbb{\tilde{E}}$ is the expectation taken with respect to $\mathbb{\tilde{P}}$.

The blue part can be written as

$$
\tilde{\mathbb{E}}\bigg(\overline{\overline{c\cdot\partial_\nu\Phi}}(\,\bar{Y}_t,\mathcal{L}_{\,\bar{Y}_t})(\,\tilde{\bar{Y}}_t)\bigg)=\int_{\mathbb{R}^{d_2}}\overline{\overline{c\cdot\partial_\nu\Phi}}(\,\bar{Y}_t,\mathcal{L}_{\,\bar{Y}_t})(\tilde{y})\mathcal{L}_{\,\bar{Y}_t}(\mathrm{d}\tilde{y}),
$$

which depends on \bar{Y}_t and its distribution $\mathcal{L}_{\bar{Y}_t}.$

The new drift and the diffusion coefficients are given by

result
\new drift and the diffusion coefficients are given by
\n
$$
\overline{H \cdot \nabla_y \Phi}(y, \mu) := \int_{\mathbb{R}^{d_1}} H(x, \zeta^{\mu}, y, \mu) \cdot \nabla_x \Phi(x, \zeta^{\mu}, y, \mu) \zeta^{\mu}(\mathrm{d}x),
$$
\n
$$
\overline{c \cdot \nabla_x \Phi}(y, \mu) := \int_{\mathbb{R}^{d_1}} c(x, \zeta^{\mu}, y, \mu) \cdot \nabla_y \Phi(x, \zeta^{\mu}, y, \mu) \zeta^{\mu}(\mathrm{d}x),
$$
\n
$$
\overline{H \cdot \Phi}(y, \mu) := \int_{\mathbb{R}^{d_1}} H(x, \zeta^{\mu}, y, \mu) \cdot \Phi(x, \zeta^{\mu}, y, \mu) \zeta^{\mu}(\mathrm{d}x),
$$
\n
$$
\overline{\overline{\cdots \partial_\nu \Phi}}(y, \mu)(\tilde{y}) := \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_1}} c(\tilde{x}, \zeta^{\mu}, \tilde{y}, \mu) \cdot \partial_\nu \Phi(x, \zeta^{\mu}, y, \mu)(\tilde{x}) \zeta^{\mu}(\mathrm{d}\tilde{x}) \zeta^{\mu}(\mathrm{d}x)
$$
\n
$$
\overline{\phi_{\nu} \Phi}(x, \zeta^{\mu}, y, \mu)(\tilde{x}) \zeta^{\mu}(\mathrm{d}\tilde{x}) \zeta^{\mu}(\mathrm{d}x)
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$$
\overline{\phi_{\nu} \Phi}(x, \zeta^{\mu}, y, \mu)(\tilde{x}) \zeta^{\mu}(\mathrm{d}x) \zeta^{\mu}(\mathrm{d}x)
$$
\n
$$
\overline{\phi_{\nu} \Phi}(x, \zeta^{\mu}, y, \mu)(\tilde{x}) \zeta^{\mu}(\mathrm{d}x) \zeta^{\mu}(\mathrm{d}x)
$$

and

$$
\overline{\overline{c\cdot\partial_\nu\Phi}}(y,\mu)(\tilde y):=\int_{\mathbb{R}^{d_1}}\int_{\mathbb{R}^{d_1}}c(\tilde x,\zeta^\mu,\tilde y,\mu)\\\cdot\partial_\nu\Phi(x,\zeta^\mu,y,\mu)(\tilde x)\zeta^\mu(\mathrm{d}\tilde x)\zeta^\mu(\mathrm{d} x).
$$

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Consider the Langevin equation:

work

\nthe Langevin equation:

\n
$$
dX_t = -\nabla V(X_t)dt - \alpha(X_t - \mathbb{E}X_t)dt + dW_t,
$$
\n3.11)

\n> 0 is the intensity of the mean-field interaction, and

\n
$$
V(x) = x^4/4 - x^2/2.
$$
\nExercise (JSNU)

\nPoisson equation and diffusion approximation

\nNow 27, 2022

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where $\alpha > 0$ is the intensity of the mean-field interaction, and

$$
V(x) = x^4/4 - x^2/2.
$$

 $2Q$

Consider the Langevin equation:

$$
dX_t = -\nabla V(X_t)dt - \alpha (X_t - \mathbb{E}X_t)dt + dW_t, \qquad (3.11)
$$

where $\alpha > 0$ is the intensity of the mean-field interaction, and

$$
V(x) = x^4/4 - x^2/2.
$$

\diamond Dawson (1983):

Work

the Langevin equation:
 $dX_t = -\nabla V(X_t)dt - \alpha(X_t - \mathbb{E}X_t)dt + dW_t,$

> 0 is the intensity of the mean-field interaction, and
 $V(x) = x^4/4 - x^2/2.$

n (1983):

transition: there exist three invariant measures for S[D](#page-0-0)E (3.

α Phase transition: there exist three invariant measures for SDE [\(3.11\)](#page-26-1) when α is big enough.

Consider the two-scales Langevin equation:

work

\nthe two-scales Langevin equation:

\n
$$
dX_t^{\varepsilon} = -\nabla V^{\varepsilon}(X_t^{\varepsilon})dt - \alpha(X_t^{\varepsilon} - \mathbb{E}X_t^{\varepsilon})dt + dW_t,
$$
\n
$$
V^{\varepsilon}(x) := V(x, x/\varepsilon).
$$
\nlino, Gvalani, Pavliotis (2022):

\neffect of multi-scale on the phase transition of the system.

\nFor example, the system is

\n
$$
V^{\varepsilon}(x) = V(x, x/\varepsilon).
$$
\nHere, if U is the same value of U and U is the same value of U and

where

$$
V^{\varepsilon}(x):=V(x,x/\varepsilon).
$$

Delgadino, Gvalani, Pavliotis (2022):

The effect of multi-scale on the phase transition of the system.

Consider the two-scales Langevin equation:

$$
dX_t^{\varepsilon} = -\nabla V^{\varepsilon}(X_t^{\varepsilon})dt - \alpha(X_t^{\varepsilon} - \mathbb{E}X_t^{\varepsilon})dt + dW_t, \qquad (3.12)
$$

where

$$
V^{\varepsilon}(x):=V(x,x/\varepsilon).
$$

Delgadino, Gvalani, Pavliotis (2022):

The effect of multi-scale on the phase transition of the system.

Aim:

work

the two-scales Langevin equation:
 $\mathrm{d}X_t^\varepsilon = -\nabla V^\varepsilon(X_t^\varepsilon) \mathrm{d}t - \alpha (X_t^\varepsilon - \mathbb{E}X_t^\varepsilon) \mathrm{d}t + \mathrm{d}W_t,$
 $V^\varepsilon(x) := V(x, x/\varepsilon).$

lino, Gvalani, Pavliotis (2022):

ffect of multi-scale on the phase transition of \diamond study the limit $\bar X_t$ of X_t^ε as $\varepsilon\to 0$ \circ study the combined limit of $\varepsilon \to 0$ and $t \to \infty$.

$\begin{minipage}{0.9\linewidth} \textbf{T} \textbf{hank} \textbf{You} \textbf{!} \\\\ \textbf{F} \textbf{Xie} \textbf{ (JSNU)} \textbf{Poisson equation and diffusion approximation} \textbf{Nov 27, 202} \end{minipage}$ Thank You !

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