Poisson equation on Wasserstein space and diffusion approximations for McKean-Vlasov equation

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Overview

Background

Consider the following two-time-scales stochastic system:

$$\mathrm{d}Y_t^\varepsilon = F(X_{t/\varepsilon}, Y_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d, \tag{1.1}$$

where $X = (X_t)_{t \geqslant 0}$ is an ergodic Markov process possessing a unique invariant measure $\zeta(\mathrm{d}x)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

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where $X = (X_t)_{t \geqslant 0}$ is an ergodic Markov process possessing a unique invariant measure $\zeta(\mathrm{d}x)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

▷ Y_t^{ε} (slow variable): mathematical model for a phenomenon appearing at the natural time scale;

▷ $X_{t/\varepsilon}$ (fast variable): fast random environment/effects at a faster time scale (with time order $1/\varepsilon$).

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▷ $X_{t/\varepsilon}$ (fast variable): fast random environment/effects at a faster time scale (with time order $1/\varepsilon$).

Usually, system (1.1) is difficult to deal with due to the widely separated time scales. Thus a simplified equation which governs the evolution of the system over a long time scale (as $\varepsilon \to 0$) is highly desirable.

Background

Intuitively,

$$X_{t/\varepsilon} \Rightarrow \zeta(\mathrm{d}x) \quad \text{as} \quad \varepsilon \to 0.$$

Thus, by averaging the coefficient with respect to the fast variable, the slow part Y_t^ε will converge to $\bar{Y}_t$, where

$$\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d \tag{1.2}$$

with

$$\bar{F}(y) := \int_{\mathbb{R}^d} F(x, y)\zeta(\mathrm{d}x).$$

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This theory, known as the averaging principle, was first developed by Bogolyubov (1937) for ODEs and extended to the SDEs by Khasminskii (1966).

Background

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1} b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \varepsilon^{-1/2}\mathrm{d}W_t^1, & X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \\ \mathrm{d}Y_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t^2, & Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is a small parameter.

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$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1} b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \varepsilon^{-1/2}\mathrm{d}W_t^1, & X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \\ \mathrm{d}Y_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t^2, & Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is a small parameter.

The intuitive idea for deriving a simplified equation for (1.3) is based on the observation that:

- ◇ during the fast transients, the slow variable remains "constant";
- ◇ by the time its changes become noticeable, the fast variable has almost reached its quasi-steady state.

Background - Averaging principle

◇ With the natural time scaling $t \mapsto \varepsilon t$, the process $\tilde{X}^\varepsilon_t := X^\varepsilon_{\varepsilon t}$ satisfies

$$\mathrm{d}\tilde{X}^\varepsilon_t = b(\tilde{X}^\varepsilon_t, Y^\varepsilon_{\varepsilon t})\mathrm{d}t + \mathrm{d}\tilde{W}^1_t,$$

where $\tilde{W}^1_t := \varepsilon^{-1/2} W^1_{\varepsilon t}$ is a new BM.

Background - Averaging principle

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where $\tilde{W}^1_t := \varepsilon^{-1/2} W^1_{\varepsilon t}$ is a new BM.

Thus we need to consider the auxiliary process X^y_t which satisfies the frozen equation

$$\mathrm{d}X^y_t = b(X^y_t, y)\mathrm{d}t + \mathrm{d}W^1_t, \quad X^y_0 = x \in \mathbb{R}^{d_1}.$$

Under certain recurrence conditions, the process X^y_t process a unique invariant measure $\zeta^y(\mathrm{d}x)$.

Background - Averaging principle

$\diamond$ Then by averaging the coefficients with respect to parameter in fast variable, the slow part Y_t^ε will converge to $\bar{Y}_t$, where

$$\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + \mathrm{d}W_t^2 \tag{1.4}$$

with

$$\bar{F}(y) := \int_{\mathbb{R}^{d_1}} F(x, y)\zeta^y(\mathrm{d}x).$$

Background

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1}c(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \varepsilon^{-2}b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \varepsilon^{-1}\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{-1}H(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t^2, \\ X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \quad Y_0^\varepsilon = y \in \mathbb{R}^{d_2}. \end{cases} \tag{1.5}$$

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- ◇ there exist two time scales in the fast motion X_t^ε;
- ◇ even the slow process Y_t^ε has a fast varying component, which is known to be closely related to the homogenization in PDEs.

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◇ there exist two time scales in the fast motion X_t^ε;

◇ even the slow process Y_t^ε has a fast varying component, which is known to be closely related to the homogenization in PDEs.

History results:

- Papanicolaou, Stroock and Varadhan (1976);
- Pardoux and Veretennikov (2001, 03, 05).

Background

We need to consider the frozen equation

$$\mathrm{d}X_t^y = b(X_t^y, y)\mathrm{d}t + \mathrm{d}W_t^1, \quad X_0^y = x \in \mathbb{R}^{d_1}$$

and the corresponding Poisson equation in $\mathbb{R}^{d_1}$:

$$\mathscr{L}_0(x,y)\Phi(x,y) = -H(x,y), \quad x \in \mathbb{R}^{d_1}, \tag{1.6}$$

where $y \in \mathbb{R}^{d_2}$ is a parameter, and $\mathscr{L}_0(x,y)$ is given by

$$\mathscr{L}_0(x,y) := \frac{1}{2}\Delta_x + b(x,y) \cdot \nabla_x.$$

Background

As $\varepsilon \to 0$, the slow process Y_t^ε will converge to $\bar{Y}_t$ with

$$\begin{aligned} \mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + \Big(\overline{c \cdot \nabla_x \Phi}(\bar{Y}_t) + \overline{H \cdot \nabla_y \Phi}(\bar{Y}_t)\Big)\mathrm{d}t \\ + \big(\mathbb{I} + \sqrt{\overline{H * \Phi}}(\bar{Y}_t)\big)\mathrm{d}W_t^2, \end{aligned}$$

where the new drift and the diffusion coefficients are given by

$$\begin{aligned} \overline{c \cdot \nabla_x \Phi}(y) &:= \int_{\mathbb{R}^{d_1}} c(x,y) \cdot \nabla_x \Phi(x,y) \zeta^y(\mathrm{d}x), \\ \overline{H \cdot \nabla_y \Phi}(y) &:= \int_{\mathbb{R}^{d_1}} H(x,y) \cdot \nabla_y \Phi(x,y) \zeta^y(\mathrm{d}x), \\ \overline{H * \Phi}(y) &:= \int_{\mathbb{R}^{d_1}} H(x,y) * \Phi(x,y) \zeta^y(\mathrm{d}x). \end{aligned}$$

Background

Bezemek and Spiliopoulos (2022) considered the multiscale distribution dependent SDE in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1}c(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \varepsilon^{-2}b(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \varepsilon^{-1}\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{-1}H(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + F(X_t^\varepsilon, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \mathrm{d}W_t^2, \end{cases} \tag{1.7}$$

where $\mathcal{L}_{Y_t^\varepsilon}$ is the distribution of the slow process Y_t^ε.

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where $\mathcal{L}_{Y_t^\varepsilon}$ is the distribution of the slow process Y_t^ε.

The frozen equation is still given by

$$\mathrm{d}X_t^y = b(X_t^y, y, \mu)\mathrm{d}t + \mathrm{d}W_t^1$$

and the corresponding Poisson equation in $\mathbb{R}^{d_1}$ is:

$$\mathscr{L}_0(x, y, \mu)\Phi(x, y, \mu) = -H(x, y, \mu), \tag{1.8}$$

where $(y, \mu) \in \mathbb{R}^{d_2} \times \mathscr{P}(\mathbb{R}^{d_2})$ are parameters, and $\mathscr{L}_0(x, y, \mu)$ is given by

$$\mathscr{L}_0(x, y, \mu) := \frac{1}{2}\Delta_x + b(x, y, \mu) \cdot \nabla_x.$$

Background

As $\varepsilon \to 0$, the slow process Y_t^ε will converge to $\bar{Y}_t$ with

$$\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\mathrm{d}t + \Big(\overline{c\cdot\nabla_x\Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}) + \overline{H\cdot\nabla_y\Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\Big)\mathrm{d}t + \big(\mathbb{I} + \sqrt{\overline{H*\Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\big)\mathrm{d}W_t^2,$$

where the new drift and the diffusion coefficients are given by

$$\overline{c\cdot\nabla_x\Phi}(y,\mu) := \int_{\mathbb{R}^{d_1}} c(x,y,\mu)\cdot\nabla_x\Phi(x,y,\mu)\zeta^{y,\mu}(\mathrm{d}x),$$

$$\overline{H\cdot\nabla_y\Phi}(y,\mu) := \int_{\mathbb{R}^{d_1}} H(x,y,\mu)\cdot\nabla_y\Phi(x,y,\mu)\zeta^{y,\mu}(\mathrm{d}x),$$

$$\overline{H*\Phi}(y,\mu) := \int_{\mathbb{R}^{d_1}} H(x,y,\mu)*\Phi(x,y,\mu)\zeta^{y,\mu}(\mathrm{d}x).$$

Main result

Consider the multiscale distribution dependent SDE in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1}c(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \varepsilon^{-2}b(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \varepsilon^{-1}\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{-1}H(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + F(X_t^\varepsilon, \mathcal{L}_{X_t^\varepsilon}, Y_t^\varepsilon, \mathcal{L}_{Y_t^\varepsilon})\mathrm{d}t + \mathrm{d}W_t^2, \end{cases} \tag{2.9}$$

where $\mathcal{L}_{X_t^\varepsilon}$ and $\mathcal{L}_{Y_t^\varepsilon}$ are the distribution of the processes X_t^ε and Y_t^ε, respectively.

Main result

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where $\mathcal{L}_{X_t^\varepsilon}$ and $\mathcal{L}_{Y_t^\varepsilon}$ are the distribution of the processes X_t^ε and Y_t^ε, respectively.

Now, we should consider the following frozen equation

$$\mathrm{d}X_t^\mu = b(X_t^\mu, \mathcal{L}_{X_t^\mu}, \mu)\mathrm{d}t + \mathrm{d}W_t^1,$$

which is a McKean-Vlasov type equation, and the corresponding Poisson equation in $\mathbb{R}^{d_1} \times \mathscr{P}(\mathbb{R}^{d_1})$:

$$\mathscr{L}_0(x, \nu, \mu)\Phi(x, \nu, y, \mu) = -H(x, \nu, y, \mu), \tag{2.10}$$

where the $\mathscr{L}_0(x,\nu,\mu)$ is given by

$$\begin{aligned}
&\mathscr{L}_0(x,\nu,\mu)\varphi(x,\nu)\\
&:=\frac{1}{2}\Delta_x\varphi(x,\nu)+b(x,\nu,\mu)\cdot\nabla_x\varphi(x,\nu)\\
&+\int_{\mathbb{R}^{d_2}}\Big[b(\tilde{x},\nu,\mu)\cdot\partial_\nu\varphi(x,\nu)(\tilde{x})+\frac{1}{2}\partial_{\tilde{x}}\big[\partial_\nu\varphi(x,\nu)(\tilde{x})\big]\Big]\nu(\mathrm{d}\tilde{x}).
\end{aligned}$$

The first part of the operator involves the usual derivatives in the space variable (which is standard).

The integral part contains the Lions derivative of the test function with respect to the measure argument.

Main result

As $\varepsilon \to 0$, the slow process Y_t^ε will converge to $\bar{Y}_t$ with

$$\begin{aligned}\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\mathrm{d}t + \Big(\overline{c \cdot \nabla_x \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t}) + \overline{H \cdot \nabla_y \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})\Big)\mathrm{d}t \\ + \sqrt{\mathbb{I} + \overline{H * \Phi}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})}\mathrm{d}W_t^2 \\ + \tilde{\mathbb{E}}\Big(\overline{\overline{c \cdot \partial_\nu \Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})(\tilde{\bar{Y}}_t)\Big)\mathrm{d}t,\end{aligned}$$

where $\tilde{\bar{Y}}_t$ is a copy of $\bar{Y}_t$ defined on a copy $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ of the original probability space, and $\tilde{\mathbb{E}}$ is the expectation taken with respect to $\tilde{\mathbb{P}}$.

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The blue part can be written as

$$\tilde{\mathbb{E}}\Big(\overline{\overline{c \cdot \partial_\nu \Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})(\tilde{\bar{Y}}_t)\Big) = \int_{\mathbb{R}^{d_2}} \overline{\overline{c \cdot \partial_\nu \Phi}}(\bar{Y}_t, \mathcal{L}_{\bar{Y}_t})(\tilde{y})\mathcal{L}_{\bar{Y}_t}(\mathrm{d}\tilde{y}),$$

which depends on $\bar{Y}_t$ and its distribution $\mathcal{L}_{\bar{Y}_t}$.

Main result

The new drift and the diffusion coefficients are given by

$$\overline{H \cdot \nabla_y \Phi}(y,\mu) := \int_{\mathbb{R}^{d_1}} H(x,\zeta^\mu,y,\mu)\cdot \nabla_x \Phi(x,\zeta^\mu,y,\mu)\zeta^\mu(\mathrm{d}x),$$

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and

$$\overline{\overline{c \cdot \partial_\nu \Phi}}(y,\mu)(\tilde{y}) := \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_1}} c(\tilde{x},\zeta^\mu,\tilde{y},\mu) \cdot \partial_\nu \Phi(x,\zeta^\mu,y,\mu)(\tilde{x})\zeta^\mu(\mathrm{d}\tilde{x})\zeta^\mu(\mathrm{d}x).$$

Future work

Consider the Langevin equation:

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t - \alpha(X_t - \mathbb{E}X_t)\mathrm{d}t + \mathrm{d}W_t, \tag{3.11}$$

where $\alpha > 0$ is the intensity of the mean-field interaction, and

$$V(x) = x^4/4 - x^2/2.$$

Future work

Consider the Langevin equation:

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t - \alpha(X_t - \mathbb{E}X_t)\mathrm{d}t + \mathrm{d}W_t, \qquad (3.11)$$

where $\alpha > 0$ is the intensity of the mean-field interaction, and

$$V(x) = x^4/4 - x^2/2.$$

◇ Dawson (1983):
Phase transition: there exist three invariant measures for SDE (3.11) when α is big enough.

Future work

Consider the two-scales Langevin equation:

$$\mathrm{d}X_t^\varepsilon = -\nabla V^\varepsilon(X_t^\varepsilon)\mathrm{d}t - \alpha(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t, \tag{3.12}$$

where

$$V^\varepsilon(x) := V(x, x/\varepsilon).$$

◇ Delgadino, Gvalani, Pavliotis (2022):
The effect of multi-scale on the phase transition of the system.

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Aim:

◇ study the limit $\bar{X}_t$ of X_t^ε as $\varepsilon \to 0$

◇ study the combined limit of $\varepsilon \to 0$ and $t \to \infty$.

Thank You !